ALFVÉN ION-CYCLOTRON INSTABILITY IN A MIRROR TRAP WITH HYGHLY ANISOTROPIC PLASMA

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The Alfvén ion cyclotron instability is studied for mirror-confined bi-Maxwellian highly anisotropic plasmas. In such plasmas the wave length of unstable modes is of the order of the plasma scale length. Another specific feature is that a typical ion can execute several bounce oscillations along the strongly non-uniform plasma during the time of the phase divergence between the wave and cyclotron rotation. Traditional approaches such as WKB method and local dispersion relation fail under these conditions.

An integral equation for the modes is derived. The spatial distribution of the eigenmodes as well as the marginal stability conditions are found by numerical solution of this equation. The asymptotics of these results in the limit of infinitely large anisotropy are obtained analytically. It is found that the mirror-confined highly anisotropic plasma can be much more stable than it follows from the traditionally used scaling.

I. INTRODUCTION

The Alfvén ion cyclotron (AIC) instability is electromagnetic instability which can develop in an anisotropic plasma. This instability was observed in the open traps (TMX [1], GDT end-cell [2]) and in the magnetosphere [3]. The theory of the AIC instability in the uniform plasmas was described in details in a number of papers (e.g., see [4, 5, 6]). The traditionally used stability scaling $\beta_\perp \mathcal{A}^2 < Const$ was formulated in [1, 7], where $\beta_\perp$ is the ratio of the transversal plasma pressure to magnetic field pressure in the midplane, the anisotropy factor $\mathcal{A}$ is the ratio of the transversal and longitudinal plasma pressures in the midplane and $Const \sim 8$ (it depends on the plasma sizes). The scaling was formulated on the basis of the numerical results of [8], where the AIC instability in mirror-confined bi-Maxwellian plasma was studied by using the local dispersion relation and WKB method in the range of moderate values of anisotropy.

The purpose of the present paper is to study the AIC instability in the case of highly anisotropic mirror-confined plasmas. In the section **II** some specific features of the unstable modes in such plasmas are discussed and parameters of the marginal stability are estimated. In the section **III** the integral equation for the eigenmodes is briefly derived. In the section **IV** the asymptotic spatial distribution of the eigenmodes as well as the marginal stability condition are found analytically in the limit of infinitely large anisotropy. Some particular results are presented in the section **V**.

II. ESTIMATIONS

According to the analysis given in [5] the AIC modes can be unstable if the wave frequency satisfies the condition

$$\omega < \Omega_{ci}(1 - 1/\mathcal{A}), \tag{1}$$

where Ω_{ci} is the ion cyclotron frequency. In the case of the marginal frequency $\Omega_{ci}(1-1/\mathcal{A})$, an ion, whose velocity satisfies the cyclotron resonant condition $\omega - \Omega_{ci} - kv_\parallel = 0$, moves along an isoline of the unperturbed distribution function. The condition (1) provides existence of "inverse ion population" [7] along a resonant ion orbit. According to the dispersion relation (see [5]) and Eq.(1) the corresponding wave number k has to satisfy the inequality $k^2 < \mathcal{A}\omega_{pi}^2/c^2$, where ω_{pi} is the ion plasma frequency and c is the speed of light.

On the other hand, the wave length is limited by the anisotropic plasma scale length, $1/k < l$, which is related with the magnetic field variation scale length L as $l = L/\sqrt{\mathcal{A}}$ in the case of mirror-confined bi-Maxwellian plasma. Summarizing these inequalities one finds that the instability can not arise if the following condition is satisfied,

$$\omega_{pi}^2 L^2/c^2 \equiv \beta_\perp L^2/\rho_\perp^2 \lesssim 1\,, \tag{2}$$

where $\rho_\perp = \sqrt{2T_\perp/m_i}/\Omega_{ci}$ is the ion Larmor radius.

Comparison of Eq.(2) with the traditional scaling shows that the WKB based traditional scaling dominates, when the anisotropy factor $\mathcal{A}$ is much less than the ratio $L/\rho_\perp$. In this case the unstable wave is localized between WKB turning points. In the opposite case,

$$\mathcal{A} \gg L/\rho_\perp \,, \tag{3}$$

the unstable mode is localized because of reflection by the strong inhomogeneity of the anisotropic plasma and the condition (2) approximately determines the stability threshold. (The stabilizing effect of the longitudinal nonuniformity was noted in [6]. The estimated in [6] threshold is even $1/\beta_\perp$ times greater than the estimation given by Eq.(2), because the only fluid limit of the unstable modes was considered there.)

Another specific feature of the highly anisotropic case is that a typical ion can execute several bounce oscillations along the strongly non-uniform plasma during the time of the phase divergence between the wave and cyclotron rotation. It follows from the estimation of the bounce frequency, $\Omega_b \sim \Omega_{ci}\rho_\perp/L$, which, according to Eq.(3), is greater than $|\Omega_{ci} - \omega| \sim \Omega_{ci}/\mathcal{A}$. (The closeness to the instability threshold is assumed in the last estimation).

Thus, the traditional approaches such as the WKB method and local dispersion relation fail under the condition (3) and the eigenmode problem has to be described in terms of an integral equation.

III. THE INTEGRAL EQUATION FOR THE EIGENMODES

The integral equation for the eigenmodes was derived from the linearized Vlasov-Maxwell equations in the way roughly similar to the way given in Appendix A of [8]. The use of the simplifying conditions such as $\mathcal{A} \gg L/\rho_\perp \gg 1$, $\beta_\perp \ll 1$ (close to the instability threshold) and $\rho_\perp \ll 1/k_\perp \ll r_p$ (where r_p is the plasma radius) make the equation much more applicable for a numerical solution. The assumed axisymmetry, paraxiality and quadratic z-dependence of the background field, $B \simeq B_0(1 + z^2/L^2)$, simplify analytical expressions for unperturbed ion orbits governed by the Hamiltonian $H \simeq \mu B + p_\parallel^2/(2m_i)$, where μ is the ion magnetic moment. The bi-Maxwellity of the background distribution function and the smallness of the anisotropic plasma length $l \equiv L/\sqrt{\mathcal{A}}$ allow to calculate some intermediate integrals analytically. The resulted integral equation for the flux coordinate Fourier components of the circular polarized electric field, $E_\pm = E_x \pm iE_y$, can be written in the form

$$\int_{-\infty}^{\infty} g_\pm[\breve{\omega}, \breve{k}, \breve{k}'] E_\pm[\breve{k}']\, d\breve{k}' = \lambda \left\{ (\breve{k}^2 + \breve{k}_\perp^2/2)E_\pm - (\breve{k}_\perp^2/2)E_\mp \right\}, \tag{4}$$

where $\breve{\omega} \equiv \omega/\Omega_{ci0}$ is the dimensionless complex frequency, Ω_{ci0} is the cyclotron frequency in te midplane, $\breve{k} \equiv k_\parallel l$, $\breve{k}_\perp \equiv k_\perp l$ and the eigenvalue λ for a physically meaning solution has to be pure real and equal $1/\beta_\perp$. The kernel $g_\pm$ is the follows

$$g_\pm \simeq \mathcal{A}\frac{1+|\breve{k}-\breve{k}'|}{4\breve{\rho}^2} e^{-|\breve{k}-\breve{k}'|} + \{\breve{\omega} \mp (1 - 1/\mathcal{A})\} \times \frac{\mathcal{A}}{\breve{\rho}^2} \sum_{n=0}^{\infty} \int_0^{\infty} \frac{u \breve{\mathcal{J}}_n[x_n, \breve{k}u] \breve{\mathcal{J}}_n[x_n, \breve{k}'u] h_n[X_n]}{(1+u^2)^{5/2} \breve{\Omega}_\pm[u]}\, du \,, \tag{5}$$

where $\breve{\rho} \equiv \rho_\perp/l$, $\breve{\Omega}_\pm[u] \equiv \breve{\omega} \mp (1 + u^2/(2\mathcal{A}))$, $x_n[u] \equiv \pm n u^2/(4\breve{\Omega}\mathcal{A})$, $X_n[u] \equiv \breve{\Omega}\sqrt{1+u^2}/(n\breve{\rho}\sqrt{\mathcal{A}})$, $h_0 \equiv -3/4$, $h_{n>0}[X] \equiv 2\sqrt{\pi}X^5 \exp[-X^2](i - \mathrm{erfi}\,[X]) + 2X^4 + X^2$ $\breve{\mathcal{J}}_n[x, y] = \sum_{m=-\infty}^{\infty} J_m[x] J_{n+2m}[y]$ and $J_n[x]$ are the Bessel functions. The series in Eq.(5) fast converge, therefore the integrating code selects only a moderate number of the terms to get an appropriate precision. The imaginary part of the function $h_n[X_n]$ at real X_n appears due to the Landau cyclotron resonances $\omega + n\Omega_b = \langle \Omega_{ci} \rangle$, where $\langle \Omega_{ci} \rangle$ is the bounce averaged cyclotron frequency.

A set of eigen-solutions of Eq.(4) obtained at a lattice on the plane ($\breve{k}$,$\breve{k}'$) is analyzed as a set of functions of the frequency $\breve{\omega} = \breve{\omega}_r + i\gamma$. A set of values of $\breve{\omega}_r$ resulting in vanishing of the imaginary part of some eigenvalue λ gives a discrete set of $\beta_\perp = 1/\lambda$ corresponding to the growth rate γ. (We take the lattice so large that the minimal values of $\beta_\perp$ do not depend on its details.) The minimal value of $\beta_\perp$ at $\gamma \to 0$ is the stability threshold for a chosen anisotropy factor $\mathcal{A}$. When the chosen $\mathcal{A}$ is much larger than $L/\rho_\perp$, the set of critical $\beta_\perp$ and the corresponding eigen-vectors are found to be very close to the analytically obtained solutions described in the next section.

IV. ANALYTICAL SOLUTION AT THE LIMIT OF INFINITE ANISOTROPY

In the limit $\mathcal{A} \to \infty$ the scale length of the anisotropic plasma tends to zero, the ratio $k_\perp/k_\parallel$ vanishes and the threshold eigenfrequency tends to $\Omega_{ci0}(1-1/\mathcal{A})$. Since, the local bi-Maxwellian plasma anisotropy is $\tilde{A}[z] = (1/\mathcal{A} + \Omega_{ci}[z]/\Omega_{ci0})^{-1}$, the relation $\omega/\Omega_{ci}[z] = 1 - 1/\tilde{A}[z]$ is satisfied everywhere. Consequently, Eqs.(4,4) at the threshold appear to be equivalent to the following equation in the spatial representation

$$(1+\breve{z}^2)^2\partial_{\breve{z}}^2 E_+ + \Lambda^2 E_+ = 0\,, \qquad (6)$$

where $\breve{z} \equiv z/l$, $\Lambda \equiv \omega_{pi0}L/c$, ω_{pi0} is the ion plasma frequency in the midplane and $E_+[\breve{z}]$ is the electric field component rotating in the ion direction. Note, the ion thermal motion does not play a role in the wave equation for such a wave. This effect is known even for the case of non-linear waves [9]. Since any ion moves along an isoline of the distribution function in such a wave, the effects related with Landau damping vanish.

The general solution of Eq.(6) is

$$E_+ = E_0\sqrt{1+\breve{z}^2}\,\cos\left[\sqrt{1+\Lambda^2}\arctan\left(\breve{z}\right)+\varphi_0\right], \qquad (7)$$

where E_0 and φ_0 are arbitrary constants. The corresponding magnetic field circular component is $B_+ = \frac{c}{\omega l}\partial_{\breve{z}}E_+$. The magnetic field component and the energy density $W[z] = \frac{E_+^2+B_+^2}{8\pi} + \frac{\omega_{pi}^2[z]}{(\Omega_{ci}[z]-\omega)^2}\frac{E_+^2}{8\pi}$ vanish at infinity, when the parameter Λ takes the discrete values $\Lambda = \sqrt{N(N+2)}$, where N is a positive integer. Therefore, the threshold value of $\beta_\perp = \Lambda^2\rho_\perp^2/L^2$ is minimal, when $N=1$ and $\Lambda^2 = 3$. The Figure 1 shows the localization of the wave energy and magnetic field perturbation in the plasma non-uniformity scale length. The asymptotic values of E_+ at large $|\breve{z}|$ correspond to the outgoing waves, whose energy and wavenumber tend to zero in the limit $\mathcal{A}\to\infty$ (compare with Fig. 2).

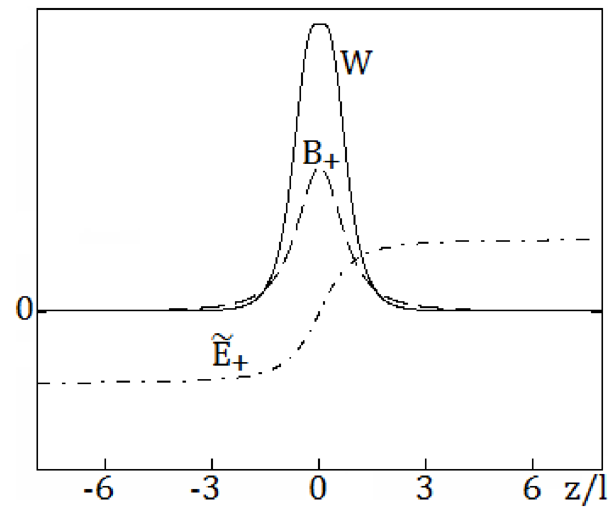

Figure 1: The wave energy density W (in arbitrary units) and field perturbations vs. z in the first eigenmode of Eq.(6). In order to compare the field perturbations the plotted electric field is normalized as $\tilde{E}_+ = E_+\,c/(2\,\Omega_{ci0}l)$

V. RESULTS

The Figure 2 shows an example of z-dependence of the field perturbations we obtained by numerical solution of the integral equation. In order to show a realistic view of the outgoing waves a small adding of Boltzmann-distributed cold isotropic plasma and finite mirror ratio $\mathcal{R}=2$ are taken into account here in contrast with Eq.(4). Besides E_+, B_+ the outgoing waves contain the same order of magnitude E_-, B_- component. So the the outgoing wave polarization is close to the linear one, while the polarization of the main central spike is closer to the circular one.

The Figure 3 shows the stability threshold data in terms of $\beta_\perp$ vs. $\mathcal{A}$ summarized from various theoretical and experimental sources. The left and right parts of the figure contain the following data.

The left part, $\mathcal{A} < L/\rho_\perp$: The lower bold curve is the absolute AIC instability threshold in uniform plasmas taken from [8]. It is continued by the thin curve presenting our numerical calculation of the absolute instability threshold. The bold curves I and II show the instability threshold for the cases $L/\rho_\perp = 50$, $r_p/\rho_\perp = 15$ and $L/\rho_\perp = 15$, $r_p/\rho_\perp = 2.7$, taken from [8]. The dashed line $\beta_\perp\mathcal{A}^2 = 3.51841$ is the asymptotic threshold of the absolute instability in uniform plasmas we obtained analytically in the limit $\mathcal{A}\to\infty$. The data of AIC stable 2XIIB experiment ($L/\rho_\perp \simeq 15$, $r_p/\rho_\perp \simeq 2.7$, $\mathcal{A}\simeq 5$, $\beta_\perp \simeq 0.33$) and AIC unstable TMX experiment ($L/\rho_\perp \simeq 33$, $r_p/\rho_\perp \simeq 7.7$, $\mathcal{A}\simeq 14$, $\beta_\perp \simeq 0.07$) are taken from [1].

The right part, $\mathcal{A} > L/\rho_\perp$: The curve is the AIC instability threshold resulted from Eq.(4) solution for the case with $k_\perp\rho_\perp \simeq 0.13$ and $L/\rho_\perp \simeq 18$. The dot-dashed line, $\omega_{pi0}^2L^2/c^2 \equiv \beta_\perp L^2/\rho_\perp^2 = 3$, shows the analytical asymptotic threshold for the mirror-confined plasmas in the limit $\mathcal{A}\to\infty$. The ellipse approximately indicates the range where the AIC instability occurred in the GDT end-cell experiment [2].

VI. SUMMARY

A linear theory of AIC instability for highly anisotropic mirror-confined bi-Maxwellian plasmas was presented. An integral equation for eigenmodes was derived and solved numerically. The numerical results are in approximate agreement with preliminary experimental results. Besides, the asymptotic stability threshold and spatial distribution of the eigenmodes were found analytically in the limit of infinitely large anisotropy. It was shown that the mirror-confined high-

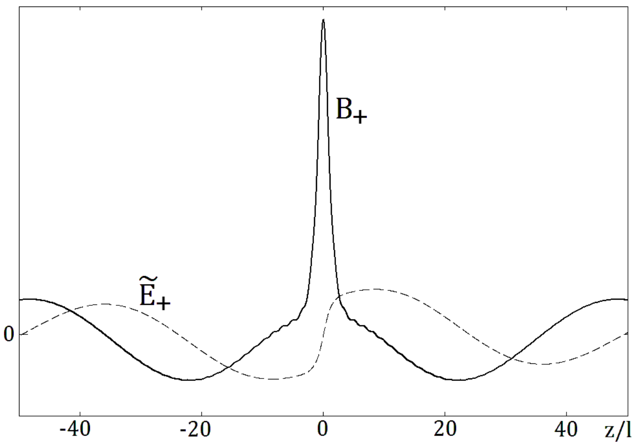

Figure 2: Example of z-dependence of the real parts of the field perturbations B_+ and E_+ resulted from the integral equation solution. The plotted electric field is normalized as $\tilde{E}_+ = E_+\, c/(10\,\Omega_{ci0} l)$. The used parameters are $\mathcal{A} = 25$, $L/\rho_\perp \simeq 0.18$, $k_\perp \rho_\perp \simeq 0.13$ and $\mathcal{I}\mathrm{m}[\omega] \simeq 10^{-4}\Omega_{ci0}$.

ly anisotropic plasma can be much more stable than it follows from the traditionally used scaling.

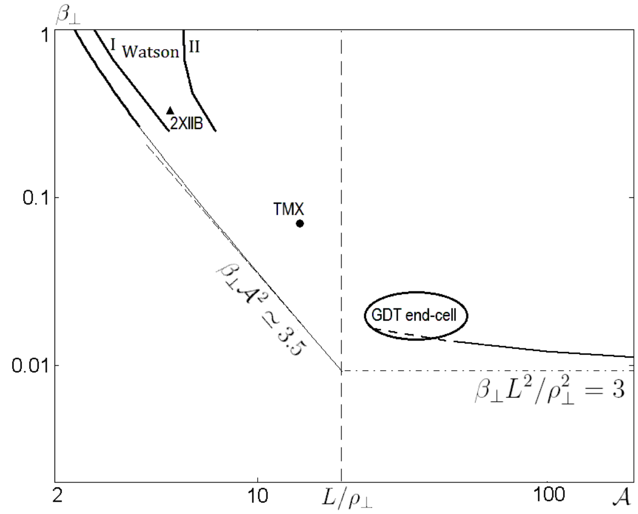

Figure 3: Summary of threshold $\beta_\perp$ vs. $\mathcal{A}$ from various sources. The bold curves in the left part present results of [8]. The curve in the right part is the threshold resulted from Eq.(4) solution. The ellipse approximately indicates the range where the AIC instability occurred in the GDT end-cell experiment [2]. Other details see in the text.

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