EQUILIBRIUM OF A HIGH-β PLASMA WITH SLOSHING IONS ABOVE THE MIRROR INSTABILITY THRESHOLD

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Within paraxial approximation it is shown that in an anisotropic plasma confined an open-ended system with β above the mirror instability threshold a "magnetic hole" is formed near the turning point of the sloshing ions.

The inequality

$$\frac{\partial}{\partial B}\left(p_{\perp}+\frac{B^2}{8\pi}\right)>0, \tag{1}$$

that guaranties stability against the mirror modes, is commonly recognized as a necessary condition for the boundary value problem of the plasma equilibrium to be well posed [1]. According to Ref. [2], it also limits the use of paraxial approximation to the case of rather low transverse plasma pressure $p_{\perp}$. Given these indications, we nevertheless assume that the paraxial approximation can still be used for calculating plasma equilibrium in linear confinement systems, even above the threshold for the mirror instability, almost in the entire bulk of the plasma column, except for some small regions. A similar situation was addressed by K. Lotov [3]. In his example, a magnetic hole with exactly zero magnetic field is formed around the axis of isotropic plasma column in a superconducting expander of an axisymmetric confinement system.

We consider a plasma configuration, typical for systems with an intense slope injection of high-energy neutral atoms, but address a much mode dense plasma as compared to earlier experimental studies. The atoms are trapped into a target, relatively cold plasma through the change exchange process, thus forming a population of fast sloshing ions. The ions bounce off the magnetic mirrors building up narrow pressure peaks at the turning points. As the pressure of sloshing ions rises in the course of injection, the criterion Eq. (1) breaks starting from the plasma axis, where the pressure is maximal.

Within standard paraxial approach we assume that the magnetic field B inside the plasma column can be related to the external field H by the reduced equation of transversal equilibrium,

$$p_{\perp}+\frac{B^2}{8\pi}=\frac{H^2}{8\pi}, \tag{2}$$

where the transversal pressure, $p_{\perp}=p_{\perp}(\Phi,B)$, is interpreted as a given function of the magnetic flux Φ and the actual magnetic field B. The approximate Eq. (2) is derived from the exact equation of transversal equilibrium,

$$\frac{\partial}{\partial n}\left(B^2+8\pi p_{\perp}\right)=\kappa\left(2B^2+8\pi p_{\perp}-8\pi p_{\parallel}\right), \tag{3}$$

by dropping the right-hand side, proportional to the field line curvature κ, and integrating the left-hand side over the direction of normal n to the magnetic field line. In paraxial approximation, the curvature is assumed to be small, and the vacuum field $H=H(z)$ a given function of the coordinate z along the axis of symmetry.

In the same approximation, the magnetic flux Φ as a function of r and z can be implicitly determined from the equation

$$\frac{\partial r^2}{\partial\Phi}=\frac{1}{\pi\,B(\Phi,H)}, \tag{4}$$

where $B(\Phi,H)$ stands for a root of Eq. (2). Outside the plasma column $B=H$ so that a non-trivial part of the problem resides in the interval $0\leqslant\Phi\leqslant\Phi_p$, where $p_{\perp}$ is distinguished from zero. For a known function $B(\Phi,H)$, Eq. (4) can be formally solved in quadratures as

$$r^2(\Phi,H)-r_p^2(H)=\int_{\Phi}^{\Phi_p}\frac{\mathrm{d}\Phi}{\pi\,B(\Phi,H)}, \tag{5}$$

where the radius of the column, $r_p(H)$, is determined from the condition $r(0,H)=0$.

The described method for solving Eqs. (2) and (4) works fine, if the dependency of B on H is one-valued. However, Eq. (2) has more than a single root regarding

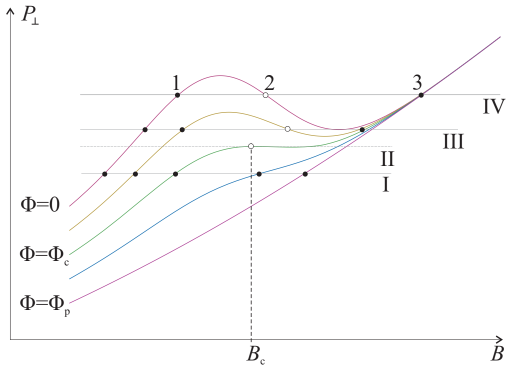

Figure 1: Graphical solution of Eq. (2). Stable and unstable roots are shown as solid and empty circles, respectively.

B, if the condition Eq. (1) breaks and the total pressure $P_\perp(\Phi, B) = B^2/8\pi + p_\perp(\Phi, B)$ as function of B becomes non-monotonic. That is what happens near the turning point of sloshing ions.

Figure 1 shows that $P_\perp(\Phi, B)$ first becomes non-monotonic function of B near the plasma axis, $\Phi = \Phi_p$, where $p_\perp$ assumes maximum, but not near the radial edge of the plasma column, $\Phi = 0$, where $p_\perp = 0$. There can exist either one or three real roots of the reduced equation of transversal equilibrium. They are the intersection points of the curve $P_\perp(\Phi, B)$ at a given Φ with a horizontal lines, I…IV, that designate the right-hand side of Eq. (2) for different values of H.

The case of single root is trivial. It corresponds to a well studied plasma equilibrium below the mirror instability threshold.

The case of three roots obviously resembles the hysteresis in ferromagnetic materials. One can choose therefore either the minimal root, labeled with the digit 1, or the maximal root 3, since the intermediate root 2 is unstable. There are some additional reasonings showing that the maximal root is most probable in the case under investigation; see Ref. [4].

When a bifurcation from a state with a single root to the state with three roots occurs, the magnetic field B experiences a sudden jump. The jump is forced when the maximal root 3 merges with the intermediate root 2, and they both disappear. At first, the jump occurs for the uppermost curve that correspond to $\Phi = 0$ (line III). However for a smaller H (i.e., for a lower line) a similar jump occurs also at $\Phi > 0$. Its magnitude gradually diminishes and the jump finally dissolves at some critical value of the magnetic flux Φ_c and B_c (line II). In the critical point,

$$\frac{\partial P_\perp}{\partial B} = 0, \qquad \frac{\partial^2 P_\perp}{\partial B^2} = 0. \tag{6}$$

Since $p_\perp + B^2/8\pi$ is constant across the plasma column, the jump in B is accompanied by a jump in $p_\perp$. The magnetic field is smaller and the plasma pressure is larger at the inner side of the surface of discontinuity. Thus, a magnetic hole, filled with dense plasma, is formed near the system axis, if the stability criterion against the mirror modes (1) is broken in the vicinity of the turning point.

The parameter Φ_c determines the radial size of the magnetic hole, which is roughly proportional to $\sqrt{\Phi_c}$. In the rest of the paper we consider the case of *shallow* magnetic hole where $\Phi_c \ll \Phi_p$, and, hence, the hole radius r_c is also small as compared with the plasma radius, $r_c \ll r_p$.

We assume that the sloshing ions have a narrow angular distribution with the angular width $\Delta\theta \ll 1$ around the angle of injection θ_*. Introducing the mirror ratio $b = B/B_0$ as the ratio of B to the magnetic field B_0 at the location of the injection in the minimum of magnetic field, one can show that the transversal pressure $p_\perp$ is peaked near the turning point located at the mirror ratio $b \approx b_* = 1/\sin^2\theta_*$. Near the maximal value, $p_{\perp *}$, the transversal pressure as a function of B varies on the scale of $\Delta B = B_0\,\Delta b$, where $\Delta b = 2\sqrt{b_* - 1}\,b_*\,\Delta\theta$ (see, e.g., [5]). Evaluating maximal negative value of $\partial p_\perp/\partial B$ as $-p_{\perp *}/\Delta B$, we readily find that the condition Eq. (1) breaks if

$$p_{\perp *} > p_{\perp c} \sim B_*\,\Delta B/4\pi. \tag{7}$$

The corresponding critical value $\beta_c \sim \Delta b/b_*$ of the parameter $\beta = 8\pi p_{\perp *}/B_*^2$ is small since $\Delta\theta \ll 1$.

In a typical plasma configuration, $p_\perp$ is monotonically decreasing function of Φ, which is maximal at $\Phi = 0$. When $p_\perp$ at $\Phi = 0$ slightly exceeds the critical value $p_{\perp c}$, the variation of the magnetic field on the size of the hole is small as compared with ΔB. This justifies expansion of the function $P_\perp(\Phi, B)$ to the Taylor series around the critical values Φ_c and B_c. Putting the expansion to the left-hand side of Eq. (2) yields

$$\psi\frac{\partial P_\perp}{\partial\psi} + \psi\,\zeta\frac{\partial^2 P_\perp}{\partial\zeta\partial\psi} + \frac{\zeta^3}{6}\frac{\partial^3 P_\perp}{\partial\zeta^3} = \frac{H^2(z)}{8\pi} - P_\perp, \tag{8}$$

where $\psi = (\Phi_c - \Phi)/\Phi_p$, $\zeta = (B - B_c)/\Delta B$, and the function $P_\perp$ and its derivatives are evaluated at $\psi = \zeta = 0$. By order of magnitude, all the derivatives of $P_\perp$ are equal to $p_{\perp *}$ with minor refinement that $\partial^2 P_\perp/\partial\zeta\,\partial\psi$ is negative. Being a rising function of z, the right-hand side of Eq. (8) passes through zero somewhere near the turning point. Let it occurs at $z = 0$. Expanding the right-hand side around $z = 0$, we can write it in the form $\frac{1}{8\pi}H_c^2\,z/L$, where L stands for the gradient length of the vacuum magnetic field, and H_c is the vacuum magnetic field in the plane $z = 0$.

Dividing Eq. (8) by $\partial^3 P_\perp/\partial\psi^3$ finally yields

$$\tfrac{1}{6}\zeta^3 - \alpha\psi\,\zeta = z/\ell - \gamma\psi, \tag{9}$$

where

$$\alpha = -\frac{\partial^2 P_\perp}{\partial\zeta\partial\psi}\;\Big/\;\frac{\partial^3 P_\perp}{\partial\zeta^3}, \qquad \gamma = \frac{\partial P_\perp}{\partial\psi}\;\Big/\;\frac{\partial^3 P_\perp}{\partial\zeta^3},$$
$$\ell = \left(\frac{\partial^3 P_\perp}{\partial\zeta^3}\;\Big/\;\frac{1}{8\pi}H_c^2\right)L.$$

Note that $\ell \sim \beta_c L$, and one of the coefficients α and γ can be equated to 1 by renormalization of the parameter ψ; using this opportunity, we set $\gamma = 1$ below.

The parameter ψ can take negative or positive values not exceeding $\psi_c = \Phi_c/\Phi_p$. For $\psi > 0$, Eq. (9) has 3 real roots, if its right-hand side falls in the range from $-(2\alpha\psi)^{3/2}/3$ to $(2\alpha\psi)^{3/2}/3$ near zero, otherwise it has a single real root. We assume that multiple roots are numbered in ascending order so that ζ_1 designates a minimal real root whereas ζ_3 stands for the maximal one. By continuity, we keep these notations for single roots matching corresponding multiple roots. Thus, there remains a single root ζ_1 if ζ_2 merges with ζ_3, and there remains ζ_3 if ζ_2 merges with ζ_1. The roots are arranged so that

$$\zeta_1 \leqslant -\sqrt{2\alpha\psi} \leqslant \zeta_2 \leqslant \sqrt{2\alpha\psi} \leqslant \zeta_3,$$

where the equality occurs if ζ_2 merges either with ζ_1 or with ζ_3.

Following the logic, described above, the intermediate root ζ_2 should never be chosen, and a "regular" solution $\zeta(\psi, z)$ is glued from ζ_3 and a part of ζ_1. The regular solution jumps from $\zeta_1 = -2\sqrt{2\alpha\psi}$ on the "internal" side of the surface discontinuity to $\zeta_3 = \sqrt{2\alpha\psi}$ at the "outer" side of the surface. Accordingly, the magnetic field jumps by the quantity of $3\sqrt{2\alpha\psi}\,\Delta B$.

At the surface of discontinuity, the left-hand side of Eq. (9) takes the value $-\frac{1}{3}(2\alpha\psi)^{3/2}$. Consequently, the position of the discontinuity in the coordinates (ψ, z) is computed from the equation

$$z/\ell = \psi - \tfrac{1}{3}(2\alpha\psi)^{3/2}. \tag{10}$$

Since $\zeta \sim \sqrt{\psi}$, ψ and z must be considered as small quantities of second order if ζ is a small quantity of the first order. With this in mind, the expression $\psi = (z/\ell - \zeta^3/6)/(1 - \alpha\zeta)$, which follows from Eq. (9), can be expanded to the 3rd order to yield an alternative equation,

$$\tfrac{1}{6}\zeta^3 - \alpha\zeta z/\ell = z/\ell - \psi. \tag{11}$$

Within desired accuracy, it is equivalent to Eq. (9) but is more convenient for graphical study, if one mentally moves in a radius on a plane of constant z. On the contrary, Eq. (9) is more appropriate, when one moves along a magnetic field line at a fixed ψ. Note that Eq. (11) has multiple roots if $z > 0$ and

$$\zeta_1 \leqslant -\sqrt{2\alpha z/\ell} \leqslant \zeta_2 \leqslant \sqrt{2\alpha z/\ell} \leqslant \zeta_3.$$

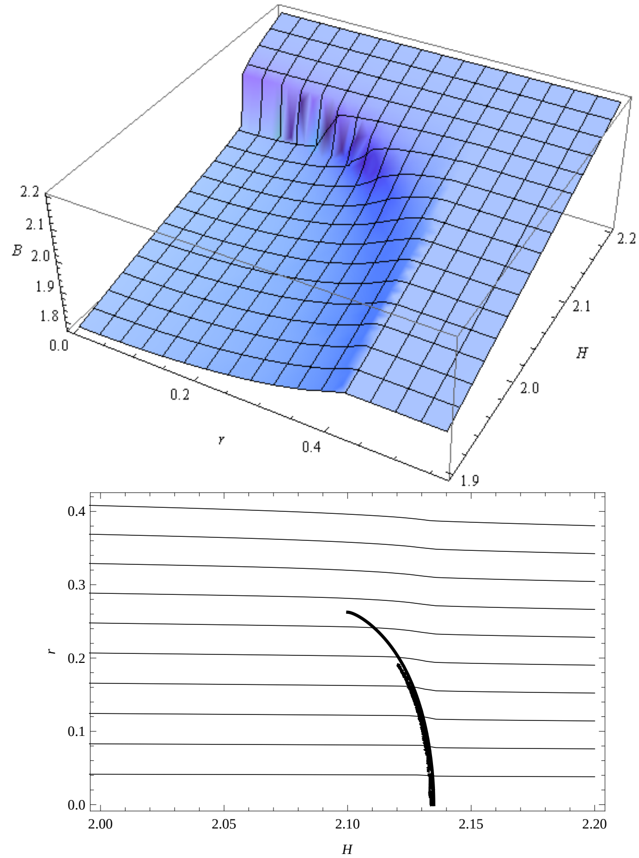

Figure 2: Magnetic hole near the turning point of sloshing ions: B as function of r and H (top), magnetic field lines and position of the jump (bottom); $\beta = 0.1$ at $b = 1$, $b_* = 2$, $\Delta b = 0.1$, $\Phi_p = 1$, $B_0 = 1$.

Regular solution ζ, fused from ζ_1 and ζ_3, monotonically rises radially from the the plasma axis toward its periphery, and axially from the system midplane toward the magnetic mirror. It means that no local mirror trap appears in the plasma as it occurs in the result of mirror instability development in a homogeneous magnetic field. In this particular sense, the inhomogeneity stabilizes the mirror instability.

Figure 2 shows a *deep* magnetic hole near the turning point of sloshing ions for a special case when all ions have same energy and their angular distribution is described by the Gaussian function with the angular width such that $\Delta b = 0.1$ and the turning point is located at $b_* = 2$.

To find the shape of the magnetic field lines in the coordinates (r, z) near the *shallow* magnetic hole, we put $B = B_0[b_c + \Delta b\,\zeta]$ to the right-hand side of Eq. (4) and write down it in the form

$$\frac{\partial r^2}{\partial \psi} = -a^2 + a^2 \frac{\Delta b}{b_c}\,\zeta, \tag{12}$$

where $a = \sqrt{\Phi_p/\pi B_0 b_c} \sim r_p$. A formal integrations with the boundary condition $r^2 = 0$ at $\psi = \psi_c$ gives

$$r^2 = a^2\left[\psi_c - \psi\right] + a^2 \frac{\Delta b}{b_c} \int_{\psi_c}^{\psi} \zeta(\psi, z)\ \mathrm{d}\psi. \tag{13}$$

Since $\zeta(\psi, z)$ experiences a jump, the function $r(\psi, z)$ has a kink at the discontinuity surface. Hence, the slope angle of the field line to the system axis, $\partial r/\partial z$, breaks there.

The integral in Eq. (13) can be computed with the aid of Eq. (11). The result of computation,

$$\int_{\psi_c}^{\psi} \zeta(\psi, z)\ \mathrm{d}\psi = \left(\frac{\alpha z}{2\ell} - \frac{\zeta^2}{8}\right)\zeta^2 + F(z),$$

involves a function $F(z)$ to be determined separately for the regions inside and outside the hole, where $\zeta = \zeta_1$ and $\zeta = \zeta_3$, respectively.

The two expressions for $F(z)$ are determined from the condition that the entire integral is zero for $\psi = \psi_c$ and is also continuous at the surface of discontinuity. Simple calculations give

$$r^2 = a^2\left[\psi_c - \psi\right] + a^2 \frac{\Delta b}{b_c}\left[\frac{\alpha z}{2\ell}\left(\zeta_1^2 - \zeta_{1c}^2\right) - \frac{1}{8}\left(\zeta_1^4 - \zeta_{1c}^4\right)\right], \tag{14}$$

inside the hole, and

$$r^2 = a^2\left[\psi_c - \psi\right] + a^2 \frac{\Delta b}{b_c} \times \left[\frac{\alpha z}{2\ell}\left(\zeta_3^2 - \zeta_{1c}^2\right) - \frac{1}{8}\left(\zeta_3^4 - \zeta_{1c}^4\right) - \frac{9}{2}\frac{\alpha^2 z^2}{\ell^2}\right] \tag{15}$$

outside it; here $\zeta_{1c} = \zeta_1(\psi_c, z)$. The shape of the hole is found by substituting $\zeta_1 = -2(2\alpha z/\ell)^{1/2}$ to Eq. (14) or $\zeta_3 = (2\alpha z/\ell)^{1/2}$ to Eq. (15). Excluding ψ and ψ_c with the aid of Eq. (9) gives

$$r_h^2 = a^2 \frac{(z_c - z)}{\ell} + \frac{2^{3/2}\alpha^{3/2}a^2}{3}\frac{\left(z_c^{3/2} - z^{3/2}\right)}{\ell^{3/2}} - \mathcal{O}(\Delta b/b_c), \tag{16}$$

where z_c is the coordinate where the surface of discontinuity intersects the axis z; it is determined from the equation

$$\psi_c = z_c/\ell + \tfrac{1}{3}\left(2\alpha z_c/\ell\right)^{3/2}, \tag{17}$$

which follows from Eq. (13) at $\psi = \psi_c$, $z = z_c$, and $\zeta = -2\left(2 z_c/\ell\right)^{1/2}$. The discontinuity surface has approximately the shape of a truncated paraboloid of revolution that terminates at the plane $z = 0$; thus, z_c is the length of the hole, and $r_c = a\sqrt{\alpha z_c/\ell}$ is its radius.

The validity of the paraxial approximation is limited by two requirements.

First, it is necessary for the slope of the magnetic field lines to be small, $|\partial r/\partial z| \ll 1$. The derivative $\partial r/\partial z$ is formally infinite at $z \to 0$, near the nose of the paraboloid, but it becomes sufficiently small if

$$z_c - z \gg z_c\,(a/L)^2\,. \tag{18}$$

Second, it is necessary to validate the use of approximate equality Eq. (2) instead of the exact Eq. (3). Since the derivative $\partial r/\partial z$ breaks on the surface of discontinuity, the curvature κ contains a delta function, $\kappa = \{\partial r/\partial z\}\,\delta(z - z_h)$, where $\{\partial r/\partial z\}$ denotes the magnitude of the jump, and $z_h = z_c - (r_h/a)^2\,\ell$ its coordinate. Hence, $P_\perp$ has a jump on the surface of discontinuity, according to the relation

$$\left\{\frac{B^2 + 8\pi p_\perp}{2B^2 + 8\pi p_\perp - 8\pi p_\parallel}\right\} = \int_{r_h(z)-0}^{r_h(z)+0} \kappa\, dr$$

that follows from (3), whereas we assumed that it is continuous. However our calculations remain correct, provided that the relative change of $P_\perp$ is small as compared with the relative jump of the magnetic field,

$$\int_{r_h(z)-0}^{r_h(z)+0} \kappa\, dr \ll 3\,(\Delta b/b_c)\,\sqrt{2\alpha z\ell},$$

which leads to the condition

$$z_c - z \gg a^2/4\ell, \tag{19}$$

more stringent than Eq. (18). As in the example of K. Lotov [3], the paraxial approximation is violated in a small vicinity of the point $z = z_c$, where the surface of discontinuity crosses the axis z.

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