

ABOUT A ROLE OF MEAN CURVATURE IN MAGNETIC MIRROR GEOMETRY

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The mean curvature approach is considered for arbitrary magnetic field vector and magnetic surface. It is shown that the magnetic field module variation along a field line defines the mean curvature of this vector. Magnetic surfaces with constant mean curvature can be optimal for plasma confinement in strait open-ended traps and linked multimirror configurations.

The mean curvature is one of the basic notions in the geometry of surfaces and vector fields^{1,2}, while the geometry of magnetic fields, as applied to the problems of fusion plasma confinement, does not in fact invoke this notion (as is confirmed in Ref. 3). The object of the present note is to bridge this gap.

I. MEAN CURVATURE OF SURFACES AND VECTOR FIELDS

The mean curvature of a surface is a local quantity defined by the formula

$$H_S = -\frac{1}{2}(k_{\min} + k_{\max}), \quad (1)$$

where k_{\min} and k_{\max} are the minimum and maximum curvatures of the lines of intersection of the surface with mutually perpendicular planes passing through the normal to the surface at a given point. For a unit vector field \mathbf{e} with $\mathbf{e} \cdot (\nabla \times \mathbf{e}) = 0$, there exists the family of orthogonal surfaces. The mean curvature of such a vector field \mathbf{e} is called as the mean curvature of these surfaces. In the general case $\mathbf{e} \cdot (\nabla \times \mathbf{e}) \neq 0$, the mean curvature H of a vector field \mathbf{e} is defined by the formula²

$$H = -\frac{1}{2} \nabla \cdot \mathbf{e}. \quad (2)$$

For a unit vector $\mathbf{e} = \mathbf{n}$ along the normal \mathbf{n} to a surface, the general definition of the curvature H coincides with that of H_S (see definition (1)). The surface of zero mean curvature $H_S = 0$ is called a minimal surface. Among the minimal surfaces are, e.g., the

surfaces of minimum area S with a fixed boundary (soap films). Closed minimal surfaces do not exist. Among surfaces of constant mean curvature $H_S = \text{const}$ are soap films between media at different pressures p , in which case the mean curvature is just the pressure difference. Another example of the surfaces of constant mean curvature $H_S = \text{const}$ are those among the surfaces bounding regions of given volume V that have a minimum area S . Such surfaces are called isoperimetric profiles and are perfect spheres of constant curvature. The Aleksandrov theorem⁴ states that such spheres are the only nested (non-self-intersecting) closed surfaces of constant mean curvature. A consequence of the theorem is, in particular, that nested tori of constant mean curvature do not exist.

Assuming that the axis of revolution is the Z axis of a cylindrical coordinate system, we demonstrate the existence of unlimited surfaces of constant mean curvature. The generating contours of these surfaces of revolution are described by a focus of a hyperbola or an ellipse (see Fig. 1) rolled along the straight axis of revolution⁵ (rolling of a hyperbola determines one period of continuous curve (a) in Fig. 1).

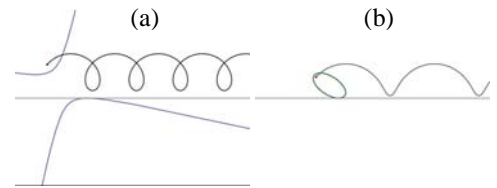


Fig. 1. Rolling of hyperbola (a) and ellipse (b)

A cylinder is a limiting case of rolling of a circle. This goes to show the existence of surfaces with $H_S = \text{const}$ that are periodic along the axis of revolution (Fig. 2 corresponds to Fig. 1a and Fig. 3 corresponds to Fig. 1b).

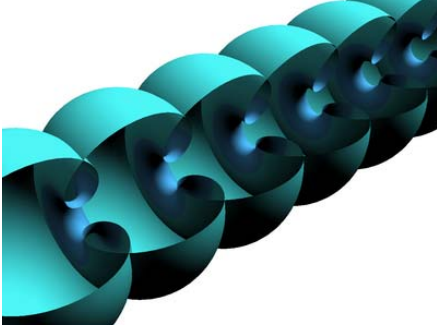


Fig. 2. Nodoid (a) — $H_S = \text{const}$ surface.

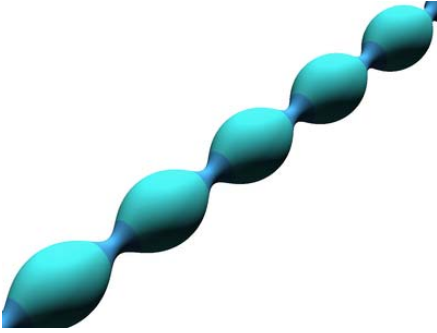


Fig. 3. Unduloid (b) — $H_S = \text{const}$ surface.

II. MEAN CURVATURE OF A MAGNETIC FIELD

According to the definition (2), a magnetic field $\mathbf{B} = B \cdot \mathbf{b}$ (where $|\mathbf{b}| = 1$ and $B = |\mathbf{B}|$ is the absolute value of the field) can be characterized by the mean curvature

$$H_b = \mathbf{b} \cdot \nabla \ln \sqrt{B}. \quad (3)$$

The variation of the absolute value of the magnetic field along the field lines is an important parameter of a plasma magnetic confinement systems and often used in the geometry of magnetic fields. For a vacuum magnetic field $\mathbf{B} = -\nabla \varphi$, where φ is the scalar magnetic potential, the mean curvature H_b coincides with that of an equipotential surface $\varphi = \text{const}$ for which the vector \mathbf{b} is a unit normal vector. Note that, in accordance with (2) the relation (3) for the mean curvature is also valid for $\nabla \times \mathbf{B} \neq 0$.

At the extreme points of the absolute value of the magnetic field along its lines, we have $H_b = 0$. In the so-called isodynamic toroidal configurations revealed by D. Palumbo⁶, the absolute value of the magnetic field is constant on the nested *magnetic* surfaces, so $H_b = 0$ over the entire confinement region. Such isodynamic toroidal configurations are possible only in the presence

of the discharge current⁷. Without the discharge current, i.e., in vacuum, the equipotential surfaces in an isodynamic configuration should be minimal, which is impossible, however, in view of the results obtained by Palumbo⁷.

The magnetic fields that form a family of nested magnetic surfaces, $\mathbf{B} \cdot \mathbf{n} = 0$, in a finite spatial region play a governing role in plasma confinement. Magnetic configurations for plasma confinement can be divided into open configurations with rippled cylindrical nested magnetic surfaces and closed configurations with nested toroidal magnetic surfaces of complicated shape. The solenoidal nature of the magnetic field implies that the toroidal magnetic flux Φ within a magnetic surface is conserved. This is why the most general equation for a family of nested magnetic surfaces is formulated in terms of the toroidal magnetic flux: $\Phi(x, y, z) = \text{const}$. The function Φ is a single-valued solution, if there is any, to the equation $\mathbf{B} \cdot \nabla \Phi = 0$ with a known magnetic field having a nonzero rotational transform. Hence, at each point of the plasma confinement region, the vector field of unit vectors normal to the magnetic surfaces, $\mathbf{n} = \nabla \Phi / |\nabla \Phi|$ such that $\mathbf{n} \cdot (\nabla \times \mathbf{n}) = 0$, is usually defined. By substituting the vector \mathbf{n} into Def. 2, it is possible to determine the mean curvature H_S of the magnetic surfaces — a quantity that plays an important role in the theory of plasma confinement systems.

II.A. Magnetic Surfaces with $H_S = \text{const}$

The energy and particle confinement in magnetic systems is commonly characterized by integral confinement times. For definiteness, let us consider the particle confinement time τ_N . This time is calculated from the formula $\tau_N = N/I$, where N is the total number of particles in the system after the injection of the particle current I . In turn, the total number of particles is $N = nV$, where n is the mean particle density in the system and V is its volume. Since, in a steady state, the injection current is equal to the loss current, we have $I = nvS$, where v is the mean velocity with which the particles escape from the system through a boundary region of area S . As a result, the particle confinement time is given by the formula $\tau_N = V/vS$. The better is the confinement, the longer is this time. Under the assumption that v is *independent* on the plasma boundary geometry, we arrive at the conclusion that the boundary surfaces of constant mean curvature are optimum for plasma confinement.

As shown in Section I, there exist periodic axisymmetric unduloid surfaces with $H_S = \text{const}$ (see Fig. 3) that can be used to optimize plasma confinement in ambipolar open magnetic systems⁸. The central solenoid of such configurations with a straight magnetic axis is characterized by the length L and on-axis mirror ratio $R = B_{\text{max}}/B_{\text{min}}$, or the ratio of the maximum and minimum radii, $r_{\text{max}}/r_{\text{min}} = \sqrt{R}$. The values of L and R determine the parameters of an ellipse that generates an optimum unduloid for a given geometry of the magnetic configuration.

In Section I, it was pointed out that tori with $H_S = \text{const}$ do not exist. It is, however, for an ambipolar open magnetic confinement system that we consider asymptotic surfaces of constant mean curvature. Fig. 4 shows a way how to close open systems by means of the magnetic surface of a Kadomtsev rippled confinement system⁹.

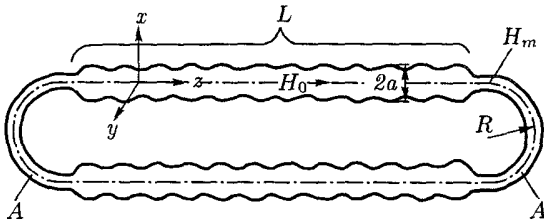


Fig. 4. Kadomtsev rippled confinement system

Let us modify the Kadomtsev system as follows. On two long straight portions, it is possible to use unduloids with a relatively weak mean magnetic field, and curvilinear elements can be half-tori with a magnetic field strong enough for the radius of the tori to be small. The system can also be arranged to have a square shape: four straight portions with unduloids are closed by curvilinear elements in the form of quarter-tori. This method of modifying the Kadomtsev system can be repeated over and over again. The main idea is that, for a sufficiently large ratio L/L_c (where L_c is the length of the curvilinear elements) and a sufficiently large mirror ratio $R = B_c/B$ (where B_c is the magnetic field in a curvilinear element), the ratio V/S is asymptotically determined only by the straight portions of the system.

III. CONCLUSIONS

The mean curvature of the magnetic field vector is related to the variation of the absolute value of the magnetic field along its lines. In the presence of magnetic

surfaces and, consequently, of the orthonormalized magnetic basis $\mathbf{b}, \mathbf{n}, \mathbf{t} = \mathbf{b} \times \mathbf{n}$, the mean curvature can be introduced for each basis vector. The mean curvature of the normal vector coincides with that of the magnetic surface. Magnetic surfaces of constant mean curvature, having a minimum surface area at a fixed volume, are optimum for plasma confinement in multimirror open systems and rippled tori with straight portions. All this goes to show that it may be helpful to use the notion of the mean curvature in the geometry of magnetic fields in plasma magnetic confinement systems.

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